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JACOBI OPERATORS ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

By

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Abstract. In this paper, we investigate a real hypersurface of a complex projective space CP^n in terms of the Jacobi operators. We give a local structure theorem of a real hypersurface of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$, where $R_\xi = R(\cdot, \xi)\xi$ is the Jacobi operator with respect to ξ and k is a function. Further, we classify real hypersurfaces of CP^n satisfying $\phi R_\xi = R_\xi \phi$ under the condition that $A\xi$ is a principal curvature vector. Also, we show that a complex projective space does not admit a locally symmetric real hypersurface.

0. Introduction

Let $CP^n = (CP^n, J, \tilde{g})$ be an n -dimensional complex projective space with Fubini-Study metric \tilde{g} of constant holomorphic sectional curvature 4, and let M be an orientable real hypersurface of CP^n and N be a unit normal vector on M . Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kählerian structure (J, \tilde{g}) of CP^n (see section 1). One of the typical examples of M is a geodesic hypersphere. R. Takagi ([10]) classified homogeneous real hypersurfaces of CP^n by means of six model spaces of type A_1 , A_2 , B , C , D , and E , further he explicitly write down their principal curvatures and multiplicities in the table in [11]. T. E. Cecil and P. J. Ryan ([2]) extensively investigated a real hypersurface which is realized as a tube of constant radius r over a complex submanifold of CP^n on which ξ is a principal curvature vector with principal

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curvature $\alpha = 2 \cot 2r$ and the corresponding focal map $\varphi_r : M \rightarrow CP^n$ (defined by $\varphi_r(p) = \exp_p(rN)$) has constant rank.

We remark that, in particular, a homogeneous real hypersurface of type A_1 , A_2 has a lot of nice geometric properties. For example, M. Okumura ([9]) showed that a real hypersurface of CP^n is locally congruent to one of homogeneous real hypersurfaces of type A_1 , A_2 if and only if the structure vector field ξ is Killing or if and only if the structure tensor ϕ commutes with the shape operator A ($\phi A = A\phi$).

We denote by ∇ the Levi-Civita connection with respect to g . The curvature tensor field R on M is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, where X and Y are vector fields on M . We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X . Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related with the Jacobi vector fields, which are solutions of the second order differential equation (the Jacobi equation) $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ . It is well-known that the notion of Jacobi vector fields involve many important geometric properties. In section 2, particularly we show that the Jacobi operator R_ξ with respect to the structure vector field ξ of a geodesic hypersphere is represented by $R_\xi = k(I - \eta \otimes \xi)$ where I denotes the identity transformation and k is a constant. Further, we give a local structure theorem of a real hypersurface of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$ where k is a function.

In section 3, we prove that a real hypersurface of CP^n is locally congruent to one of homogeneous real hypersurfaces of type A_1 , A_2 if and only if the structure tensor ϕ commutes with the Jacobi operator R_ξ ($\phi R_\xi = R_\xi \phi$) and $A\xi$ is a principal curvature vector. In section 4, we give another characterization of homogeneous real hypersurfaces of type A_1 , A_2 by the property that (*) the structure vector field ξ is a geodesic vector field and further the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ . We easily see that the property (*) is equivalent to the condition $R'_\xi = 0$ where we denote $R'_X = (\nabla_X R)(\cdot, X)X$ for any unit vector field X . Also, in section 4 we show that CP^n does not admit a locally symmetric ($\nabla R = 0$) real hypersurface.

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

1. Preliminaries

At first, we review the fundamental facts on a real hypersurface of CP^n . Let M be a real hypersurface of CP^n and N be a unit normal vector on M . By $\tilde{\nabla}$ we

denote the Levi-Civita connection with respect to the Fubini-Study metric of CP^n . Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M , where g denotes the Riemannian metric of M induced from \tilde{g} . An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M , we put

$$(1.1) \quad JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M , that is, we have

$$(1.2) \quad \begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

From (1.2), we get

$$(1.3) \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

From the fact $\tilde{\nabla}J = 0$ and (1.1), making use of the Gauss and Weingarten formulas, we have

$$(1.4) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$$

$$(1.5) \quad \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature 4, we have the following Gauss and Codazzi equations:

$$(1.6) \quad \begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.7) \quad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.$$

We recall the following:

PROPOSITION 1 ([8]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2 ([8]). *Assume that ξ is a principal curvature vector and corresponding principal curvature α . If $AX = \lambda X$ for X orthogonal to ξ , then we have $A\phi X = (\alpha\lambda + 2)/(2\lambda - \alpha)\phi X$.*

THEOREM 1 ([9]). *Let M be a real hypersurface of CP^n . Then the followings are equivalent:*

- (i) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2 .*
- (ii) $\phi A = A\phi$.

A ruled real hypersurface of CP^n is defined by a foliated one by complex hyperplanes CP^{n-1} and its shape operator is written down in [5]. Namely,

$$(1.8) \quad \begin{aligned} A\xi &= \alpha\xi + \mu W & (\mu \neq 0), \\ AW &= \mu\xi, \\ AZ &= 0 \end{aligned}$$

for any $Z \perp \xi$, W , where W is unit vector orthogonal to ξ , α and μ are functions on M . For more details about a ruled real hypersurface of CP^n , we refer to [6]. The ϕ -holomorphic sectional curvature $H(X)$ is defined by a sectional curvature of $\text{span}\{X, \phi X\}$. Further in [5], it was proved that

THEOREM 2. *Let M be a real hypersurface of CP^n ($n \geq 3$) with constant ϕ -holomorphic sectional curvature. Then M is locally congruent to the following spaces:*

- (1) *a geodesic hypersphere (that is, a homogeneous real hypersurface which lies on a tube of radius r over a hyperplane CP^{n-1} , where $0 < r < (\pi/2)$);*
- (2) *a ruled real hypersurface;*
- (3) *a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane CP^{n-1} as a ruled hypersurface.*

We denote $\alpha = g(A\xi, \xi)$ and $\beta = g(A^2\xi, \xi)$. We define a vector field U on M by $U = \nabla_\xi \xi$. Then from (1.2) and (1.5) we easily observe that U is orthogonal to ξ and also to $A\xi$. Since $\|U\|^2 = g(U, U) = \beta - \alpha^2$, from (1.2), (1.5) and (1.9) we have at once

LEMMA 1. *The followings are equivalent:*

- (i) ξ is a geodesic vector field.
- (ii) ξ is a principal curvature vector field.
- (iii) $\beta - \alpha^2 = 0$.

2. Real hypersurfaces of CP^n satisfying $R_\xi = k(I - \eta \otimes \xi)$

For each point $p \in M$ and each unit tangent vector $X \in T_p M$, we define a self-adjoint operator R_X of $T_p M$ by $R_X = R(\cdot, X)X$. We call R_X Jacobi operator with respect to X . It is well-known that a geodesic hypersphere M of CP^n is η -umbilical, i.e., M satisfies $AX = aX + b\eta(X)\xi$ for any tangent vector field X on M , where a and b are constants on M (cf. [11]). Thus from (1.6) we have

PROPOSITION 3. *Let M be a geodesic hypersphere of CP^n . Then M satisfies $R_\xi = k(I - \eta \otimes \xi)$ where I denotes the identity transformation and k is a constant on M .*

Furthermore, we prove

THEOREM 3. *Let M be a real hypersurface of CP^n ($n \geq 3$). Suppose that M satisfies $R_\xi = k(I - \eta \otimes \xi)$, where k is a function on M .*

In case that ξ is a principal curvature vector field with the associated principal curvature $\alpha = 2 \cot 2r$ and the rank of corresponding focal map φ_r is constant, then M is locally congruent to one of the following spaces:

- (1) *a geodesic hypersphere;*
- (2) *a homogeneous tube of radius $\pi/4$ over a totally geodesic CP^l ($1 \leq l \leq n-2$);*
- (3) *a non-homogeneous tube of radius $\pi/4$ over a $\varphi_{\pi/4}(M)$ with non-zero principal curvatures $\neq \pm 1$.*

Or in case that ξ is not a principal curvature vector field, then M is locally congruent to

- (4) *a non-homogeneous real hypersurface whose shape operator A is written as*

$$A\xi = \alpha\xi + \mu W \quad (\alpha \neq 0, \mu \neq 0),$$

$$AW = \mu\xi + \nu W,$$

$$AZ = (k-1)/\alpha Z, \quad k = 1 + \alpha\nu - \mu^2$$

for any $Z \perp \xi$, W , where W is a unit vector field orthogonal to ξ , α , μ and ν are functions on M .

PROOF. From (1.6) it follows that

$$(2.1) \quad R_\xi X = R(X, \xi)\xi = X - \eta(X)\xi + \alpha AX - \eta(AX)A\xi$$

for any vector field X on M . Suppose that $R_\xi X = k(X - \eta(X)\xi)$, where k is a function. Then from (2.1) we get

$$(2.2) \quad \alpha AX = (k-1)(X - \eta(X)\xi) + \eta(AX)A\xi$$

for any vector field X on M .

First we consider the case that ξ is a principal curvature vector field, that is, $A\xi = \alpha\xi$. Then from (2.2) we get

$$(2.3) \quad \alpha AX = (k-1)X$$

for any vector field X orthogonal to ξ . Since α is constant (Proposition 1), we divide our arguments into two cases: (i) $\alpha = 0$ or (ii) $\alpha \neq 0$.

(i) $\alpha = 0$. First, from (2.1) we see that $k = 1$ and M satisfies $R_\xi X = X - \eta(X)\xi$. Since the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant, by virtue of [2] we see that M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^k ($1 \leq k \leq n-1$) or locally congruent to a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kähler submanifold \tilde{N} with non-zero principal curvatures $\neq \pm 1$. (See also [7]).

(ii) $\alpha \neq 0$. From (2.3) we see that M has at most two distinct principal curvatures. Hence Theorem 3 in [2] implies that M is locally congruent to geodesic hypersphere.

Next, we consider the case that ξ is not principal. We may assume that

$$(2.4) \quad A\xi = \alpha\xi + \mu W,$$

where W is unit and orthogonal to ξ , $\mu \neq 0$. Then from (2.2) and (2.4) we get

$$(2.5) \quad \alpha AW = \alpha\mu\xi + (\mu^2 + k - 1)W.$$

Also, from (2.2) we get

$$(2.6) \quad \alpha AZ = (k-1)Z,$$

where Z is unit and orthogonal to ξ and W . Now we prove that $\alpha \neq 0$. If $\alpha = 0$ on M , then from (2.2) we get

$$(2.7) \quad (k-1)(X - \eta(X)\xi) + \eta(AX)A\xi = 0$$

for any vector field X on M . Putting $X = U$ in (2.7), then we obtain

$$(2.8) \quad (k-1)U = 0.$$

Set $\Omega_1 = \{p \in M : k(p) = 1\}$ and $\Omega_2 = \{p \in M : k(p) \neq 1\}$. Then $M = \Omega_1 \cup \Omega_2$. If $M = \Omega_1$ (Ω_2 is empty), then from (2.7) we find $\beta = 0$ on M , and hence by applying Lemma 1 we have $A\xi = 0$ on M . If $M = \Omega_2$ (Ω_1 is empty), then from (2.8) and Lemma 1 we see that $A\xi = \alpha\xi$ on M . Or, in case that both Ω_1 and Ω_2 are non-empty, by Proposition 1 and the continuity of α yield that ξ is a principal curvature vector field on M . Hence all the cases yield a contradiction. Thus $\alpha \neq 0$ on M .

Therefore from (2.4), (2.5) and (2.6) we have our real hypersurface M of the case (4), where we have put $\mu^2 + k - 1 = \alpha v$. From (1.6) we can easily see that a real hypersurface CP^n whose shape operator written as (2.4), (2.5) and (2.6) satisfies $R_\xi X = k(X - \eta(X)\xi)$ for any vector field X on M . Also, since the structure vector field ξ on a homogeneous real hypersurface of CP^n is a principal curvature vector field ([10]), we see that a real hypersurface of this case is non-homogeneous. (Q.E.D.)

REMARK 1. From Theorem 3 and the table in [11], we see that the normal Jacobi vector field along each geodesic trajectory of ξ on a geodesic hypersphere, a homogeneous tube of radius $\pi/4$ over a totally geodesic CP^l ($1 \leq l \leq n-2$), or a non-homogeneous tube of radius $\pi/4$ of the case (3) in Theorem 3 satisfies the spherical space form type Jacobi equation, i.e., $Y'' + kY = 0$ where k is a positive constant and $'$ denotes covariant derivative along a geodesic trajectory of ξ .

The rank of A at a point p in M is called a type number and is denoted by $t(p)$. Let M be a real hypersurface of CP^n which satisfies $R_\xi X = X - \eta(X)\xi$, i.e., $k = 1$. Then from (2.2) it follows that

$$(2.9) \quad \alpha AX = \eta(AX)A\xi$$

for any vector field X on M . If there exist a point p in M such that $\alpha(p) \neq 0$, then (2.9) implies that the type number $t(p)$ at p is at most 1. It is however seen (cf. [12]) that the point p is geodesic. So it is contradiction to the assumption that $\alpha(p) \neq 0$. Thus $\alpha = 0$ on M , and hence from (2.9) $\beta = 0$ on M . Therefore by Lemma 1, we see that $A\xi = 0$ on M .

REMARK 2. The above arguments together with (1.8) in section 1 and (20) in [7] imply that a non-homogeneous real hypersurface of the case (4) is neither a ruled real hypersurface nor of the case (3) in Theorem 2. But, we do not yet know the construction of the case (4) in Theorem 3.

Due to Theorems 2, 3 and Remark 2, we characterized a geodesic hypersphere of CP^n by following

COROLLARY 1. *Let M be a real hypersurface of CP^n ($n \geq 3$) with constant ϕ -holomorphic sectional curvature. In addition that M satisfies $R_\xi = k(I - \eta \otimes \xi)$, where k is a function, then M is locally congruent to a geodesic hypersphere.*

3. Real hypersurfaces of CP^n satisfying $\phi R_\xi = R_\xi \phi$

We see that all the cases appeared in Theorem 3 satisfies $\phi R_\xi = R_\xi \phi$. In this section, we prove

THEOREM 4. *Let M be a real hypersurface of CP^n . The structure tensor ϕ commutes with the Jacobi operator R_ξ and A_ξ is principal curvature vector field on M . Then ξ is principal curvature field on M . Further assume that $\alpha = 2 \cot 2r$ and the rank of the focal map ϕ_r is constant, then M is locally congruent to one of homogeneous real hypersurfaces of type A_1 , A_2 or a non-homogeneous tube of radius $\pi/4$ of the case (3) in Theorem 3.*

PROOF. Assume that $\phi R_\xi = R_\xi \phi$ and $A^2 \xi = \lambda A \xi$. From (1.6) we get

$$(3.1) \quad \begin{aligned} R_\xi(\phi X) &= \phi X + \alpha A \phi X + g(X, U) A \xi, \\ \phi(R_\xi X) &= \phi X + \alpha \phi A X - g(A X, \xi) U. \end{aligned}$$

From (3.1) and the assumption $\phi R_\xi = R_\xi \phi$, we find

$$(3.2) \quad \alpha(\phi A - A \phi)X = g(X, U) A \xi + g(A X, \xi) U.$$

First, we prove that ξ is principal curvature vector on M . We put $X = A \xi$ in (3.2) and using the another assumption $A^2 \xi = \lambda A \xi$, then we get $\alpha A U = (\alpha \lambda - \beta) U$, and hence we have

$$(3.3) \quad \alpha A U = 0,$$

because $\beta = \alpha \lambda$. If there exists a point $p \in M$ such that $\alpha(p) = 0$, then we see that $\beta(p) = 0$, and hence by using Lemma 1, we conclude that $A \xi = 0$ at p . So, from now we discuss where α has not zero. Then from (3.3), it follows that

$$(3.4) \quad A U = 0.$$

With (3.4) we easily see that

$$g((\nabla_X A)\xi, \xi) = d\alpha(X),$$

where d denotes the exterior differential. Since $U = \phi A\xi$, from (1.4), (1.7) and (3.4) we have

$$(3.5) \quad \nabla_\xi U = \alpha A\xi - \beta\xi + \phi\nabla\alpha,$$

where $\nabla\alpha$ denotes the gradient vector of α . Differentiating (3.4) covariantly, then by using (1.7) and (3.5) we have

$$(3.6) \quad (\nabla_U A)\xi = -\phi U - \alpha A^2\xi + \beta A\xi + A\phi\nabla\alpha.$$

Also, differentiating $A^2\xi = \lambda A\xi$ covariantly along M , then together with (1.5) we have

$$(3.7) \quad \begin{aligned} g(A\xi, (\nabla_X A)Y) + g((\nabla_X A)\xi, AY) + g(\phi AX, A^2Y) \\ = d\lambda(X)g(A\xi, Y) + \lambda g((\nabla_X A)\xi, Y) + \lambda g(\phi AX, AY). \end{aligned}$$

From (1.7) and (3.7) we have

$$\begin{aligned} \eta(X)g(A\xi, \phi Y) - \eta(Y)g(A\xi, \phi X) - 2\alpha g(\phi X, Y) \\ + g((\nabla_X A)\xi, AY) - g((\nabla_Y A)\xi, AX) + g(\phi AX, A^2Y) - g(\phi AY, A^2X) \\ = d\lambda(X)g(A\xi, Y) - d\lambda(Y)g(A\xi, X) + \lambda g((\nabla_X A)\xi, Y) - \lambda g((\nabla_Y A)\xi, X) \\ + 2\lambda g(\phi AX, AY) \end{aligned}$$

for any vector fields X and Y on M . We put $X = U$ and making use of (1.7), (3.4) and (3.6), then we have

$$(3.8) \quad g((\nabla_U A)\xi, AY) = 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y).$$

Thus, from (3.6) and (3.8) we obtain

$$(3.9) \quad \begin{aligned} 2(\alpha - \lambda)g(\phi U, Y) - \eta(Y)g(U, U) + d\lambda(U)g(A\xi, Y) \\ = -g(\phi U, AY) - \alpha g(A^2\xi, AY) + \beta g(A\xi, AY) - d\alpha(\phi A^2Y). \end{aligned}$$

Putting $Y = \xi$ in (3.9), then we get

$$d(\lambda\alpha)(U) = 2(\beta - \alpha^2).$$

Also, we put $Y = A\xi$ in (3.9), we get

$$\lambda d(\lambda\alpha)(U) = (\beta - \alpha^2)(3\alpha - \lambda).$$

Thus, we have $\beta - \alpha^2 = \alpha(\lambda - \alpha) = 0$, from which using Lemma 1 we see that $A\xi = \alpha\xi$ on M .

From (3.2) and Lemma 1, we see that

$$\alpha(\phi A - A\phi)X = 0.$$

Since α is constant, by a similar way as in the proof of Theorem 3 and using Theorem 1, we have our assertions. (Q.E.D.)

REMARK 3. If we omit the condition that $A\xi$ is a principal curvature vector, then Theorem 4 is not true. In fact, if a non-homogeneous real hypersurface of the case (4) in Theorem 3 satisfies $A^2\xi = \lambda A\xi$, then we can see that $\alpha\nu - \mu^2 = 0$ where α and μ have not zero, which yields a contradiction.

4. Real hypersurfaces of CP^n satisfying $R'_X = 0$

For each point $p \in M$ and each unit tangent vector $X \in T_p M$, we define R'_X by $R'_X = (\nabla_X R)(\cdot, X)X$. Then, in particular supposing that the structure vector field ξ of M is a geodesic vector field, it is easily seen that $R'_\xi = 0$ on M if and only if the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ (cf. [1] or [3]).

Now we prove

PROPOSITION 4. *Let M be a real hypersurface of CP^n . Suppose that ξ is a geodesic vector field on M and M satisfies $R'_\xi = 0$. Then ξ is principal curvature field on M . Further assume that $\alpha = 2 \cot 2r$ and the rank of the focal map φ_r is constant, then M is locally congruent to one of homogeneous real hypersurfaces of type A_1 , A_2 or a non-homogeneous tube of radius $\pi/4$ of the case (3) in Theorem 3.*

PROOF. Assume that ξ is a geodesic vector field on M . Then by Lemma 1, we immediately see that $A\xi = \alpha\xi$. Then from (1.6), taking account of (1.4), (1.7) and Proposition 1, we get

$$\begin{aligned} R'_\xi Y &= (\nabla_\xi R)(Y, \xi)\xi = \alpha(\nabla_\xi A)Y \\ &= \alpha(\alpha\phi AY - A\phi AY + \phi Y), \end{aligned}$$

for any vector field Y on M . Thus from the hypothesis we get

$$\alpha(\alpha\phi A - A\phi A + \phi)Y = 0.$$

Assume $AY = \lambda Y$ for Y orthogonal to ξ . Then from Proposition 2 we have

$$\alpha(\lambda^2 - \alpha\lambda - 1) = 0.$$

We see that $\lambda^2 - \alpha\lambda - 1 = 0$ implies $\lambda(2\lambda - \alpha) = \alpha\lambda + 2$, that is $\lambda = (\alpha\lambda + 2)/(2\lambda - \alpha)$. From this we also see that $\phi A = A\phi$, and hence from Theorem 1 and by similar arguments as in the proof of Theorem 3 in section 2, we have our assertions. (Q.E.D.)

PROPOSITION 5. *There does not exist a real hypersurface of CP^n whose structure vector field ξ is principal curvature vector field and satisfying $R'_V = 0$ for any vector field V orthogonal to ξ .*

PROOF. From (1.6), taking account of (1.4), we get

$$(4.2) \quad (\nabla_V R)(Y, V)V = -3\{\eta(Y)g(AV, V)\phi V - g(\phi Y, V)g(AV, V)\xi\} \\ + g((\nabla_V A)V, V)AY + g(AV, V)(\nabla_V A)Y \\ - g((\nabla_V A)Y, V)AV - g(AY, V)(\nabla_V A)V$$

for any vector field Y on M and any vector field V orthogonal to ξ . Assume that $A\xi = \alpha\xi$ and suppose that M satisfies $R'_V = (\nabla_V R)(\cdot, V)V = 0$ for any vector field V orthogonal to ξ . Then of course $R'_V = (\nabla_V R)(\xi, V)V = 0$ and from (4.2)

$$(4.3) \quad -3g(AV, V)\phi V + \alpha g((\nabla_V A)V, V)\xi + g(AV, V)(\alpha\phi AV - A\phi AV) \\ - \alpha g(\phi AV, V)AV + g(A\phi AV, V)AV = 0$$

for any vector field V orthogonal to ξ . From (4.3) we easily see that $\alpha g((\nabla_V A)V, V)\xi = 0$ and have

$$(4.4) \quad -3g(AV, V)\phi V + \alpha g((\nabla_V A)V, V)\xi + g(AV, V)(\alpha\phi AV - A\phi AV) \\ + g(A\phi AV, V)AV = 0$$

Assume $AV = \lambda V$ and $g(V, V) = 1$. Then from (4.4) and Proposition 2 we have

$$(4.5) \quad \lambda\{\alpha\lambda^2 - (8 + \alpha^2)\lambda + 3\alpha\} = 0.$$

From (4.5) and Proposition 1 we see that M has at most four distinct constant principal curvatures including α . But by the table in [11] we see that $\lambda \neq 0$, and (4.5) yield a contradiction. (Q.E.D.)

It is well-known that a locally symmetric space ($\nabla R = 0$) is locally homogeneous. Thus by virtue of R. Takagi's result ([10]) and Proposition 5, we have

COROLLARY 2. *There does not exist a locally symmetric real hypersurface of CP^n .*

REMARK 4. It was proved by the second author ([4]) that there does not exist a real hypersurface M with the parallel Ricci tensor in CP^n , $n \geq 3$.

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